12 Digital Communication in the Presence of Noise

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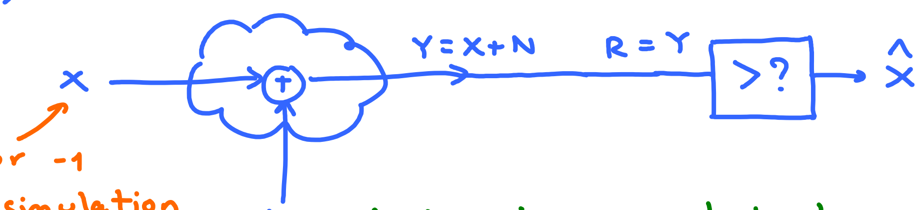
10:18 AM

Super-Simplified Model:

Here, we focus on Technique #1 in which the output of the channel is sampled directly without any processing (by the receiving filter).

We also assume that the pulse shape is designed such that there is no ISI.

In such setting, we can consider the performance of the system bit by bit:



1 or -1

(In the simulation, these two numbers are generated randomly with equal probability.)

N ← Noise is assumed to be Gaussian with expected value 0 and std. σ:

$$N \sim \mathcal{N}(0, \sigma^2)$$

At the end, the receiver try to produce $\hat{X}$ which is the same as X. (Of course, when the noise is too strong, $\hat{X}$ may $\neq X$)

To do this, it compares R with 0.

For technique #1, $R = Y$.

if $R > 0$, then $\hat{X} = 1$

$R < 0$, then $\hat{X} = -1$

Discussion: The case when $R = 0$ has zero probability of occurrence. In the case that it happens, the receiver may freely choose to guess 1/-1 or it can declare decoding error.

What we want to do now is to theoretically predict the $\underbrace{\text{performance}}$ of the system above.

By performance, we mean BER.

$P[\mathcal{E}]$
$\shortparallel$
Probability that $\hat{X} = X$.

Suppose $X = 1$ was transmitted.

Error occurs when $R < 0$

$$R < 0 \equiv X + N < 0 \equiv 1 + N < 0 \equiv N < -1 \Rightarrow P(\mathcal{E}) = P[N < -1]$$

Suppose $X = -1$ was transmitted.

Error occurs when $N > 1 \Rightarrow P(\mathcal{E}) = P[N > 1]$

For any given σ, how can we calculate $P[N<-1]$ and/or $P[N>1]$?

From ECS 315, we learn that the pdf of a (general) Gaussian RV X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

for some constants m and σ.

One can show that under such pdf, $\mathbb{E}X = m$ and $\operatorname{Var} X = \sigma^2$

To say that a RV X is described by the pdf above we usually write

$$X \sim \mathcal{N}(m, \sigma^2)$$

note that the convention is to put Var$X$ here, not the std.

Probability calculation for continuous RVs requires integration of the pdf. However, the pdf above has no close-form integral and therefore we have to use numerical integration (by, e.g., MATLAB) or use table of integration. The table is usually given in the form of the cdf of the standard Gaussian RV.

form of the cdf of the standard Gaussian RV.

$F_S(s)$ where $S \sim \mathcal{N}(0,1)$.

This cdf is usually denoted by $\Phi(s)$.

In particular,

$$\Phi(s) = F_S(s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt.$$

Given this table, our task is then to change probability calculation for general Gaussian RV into calculation involving the Φ function above.

To do this, note that for $X \sim \mathcal{N}(m, \sigma^2)$,

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{t-m}{\sigma}\right)^2} dt \underset{\tau = \frac{t-m}{\sigma}}{=} \int_{-\infty}^{(x-m)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\tau^2} d\tau = \Phi\left(\frac{x-m}{\sigma}\right)$$

Recall that our noise is $N \sim \mathcal{N}(0, \sigma^2)$.

Therefore, $F_N(n) = \Phi\left(\frac{n-0}{\sigma}\right) = \Phi\left(\frac{n}{\sigma}\right)$.

This means

$$P[N > 1] = 1 - P[N \le 1] = 1 - F_n(1) = 1 - \Phi\left(\frac{1}{\sigma}\right) = \underbrace{Q\left(\frac{1}{\sigma}\right)}_{=p},$$

and

$$P[N < -1] = \underbrace{F_N(-1)}_{=p} = \Phi\left(\frac{-1}{\sigma}\right) = Q\left(\frac{1}{\sigma}\right).$$

Look at this system as a BSC